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SPATIALLY PERIODIC SUSPENSIONS OF CONVEX PARTICLES IN LINEAR SHEAR FLOWS.

II. RHEOLOGY

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Abstract—Following the purely kinematical developments of Part I, a rigorous analysis is presented of the "almost" time-periodic low Reynolds number hydrodynamics of a spatially periodic suspension of identical convex particles in a Newtonian liquid undergoing a macroscopically homogeneous linear shear flow. By considering the case of a single particle within a unit cell of the instantaneous spatially periodic configuration, the quasistatic dynamical analysis of this infinite-particle system is effected in much the same way as for a single particle suspended in an unbounded fluid. This is accomplished via the introduction of a partitioned hydrodynamic Stokes resistance matrix, linearly relating the force, couple and stresslet on the particle in the unit cell to the translational and rotational particle-(mean) suspension slip velocities and the mean rate-of-strain of the suspension. In contrast with the unbounded fluid case for a given geometry of the individual particles, the (purely geometric) elements of the resistance matrix depend upon the instantaneous *lattice* configuration.

These dynamic quasistatic calculations for a given *instantaneous* lattice conformation, in particular that for the stresslet, are then appropriately averaged over both space and time for the class of almost time-periodic, lattice-reproducing, flows discussed in Part I. (In actually performing the time average, an important distinction is drawn between the ergodic and deterministic shear processes whose kinematical basis was laid in Part I.) In turn, this averaged dynamical information is translated into knowledge of the rheological properties of the macroscopically homogeneous suspension.

A rigorous asymptotic, lubrication-theory analysis is performed during the course of an illustrative calculation of the rheological properties of a concentrated suspension of almost-touching spheres in a simple shear flow. Contrary to the findings of a previous heuristic treatment of this same lubrication-theory problem—one that ignores evolutionary variations in the *instantaneous* geometrical configuration of the spatially periodic suspension as the shear proceeds—the time-average properties of the suspension are found to be *nonsingular* in the limit.

Finally, brief comments are offered on potential extensions of the scheme to include nonlinear phenomena, such as nonNewtonian fluids and inertial effects.

1. INTRODUCTION

Most analyses of the rheological properties of macroscopically homogeneous suspensions ignore the intrinsically time-dependent nature of the local particle-scale geometrical configuration of the suspension. This unsteady motion arises from the *relative* motion of adjacent particles suspended in the shearing motion in which they participate. Such quasisteady analyses implicitly or explicitly leap directly to some time-average spatially homogeneous representation (e.g. "random") of the mean geometrical configuration by simply ignoring the fundamental unsteadiness of the local flow field—as in "single particle" calculations in dilute suspensions (Brenner 1974). The latter approach, though perhaps correctly yielding the relative time-average spatial distribution of particle centers, fails to come to grips with the fundamental problem of how this essentially time-average kinematical information is to be coupled to the *instantaneous* quasistatic local hydrocynamic boundary-value problems posed, whose solution is prerequisite to establishing the mean suspension properties.

This conceptual dilemma clearly leaves a logical void in the further development of suspension rheology in concentrated systems, and by doing so renders suspect the rheological conclusions gleaned from such internally inconsistent analyses. This problem is resolved in the present paper, albeit in the very restricted context of perfectly ordered particle arrays

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and their temporal evolution under the influence of a shearing flow which acts to reproduce the detailed particle arrangement (almost) periodically in time.

The first paper of this series (Adler & Brenner 1985; hereafter referred to as I) was devoted to a study of the kinematics of a spatially periodic suspension of identical spherical particles, employing the geometry of numbers. This new analytical perspective will be here extended to include a rigorous calculation of the average *dynamical* properties of a suspension subjected to a macroscopic linear shear flow. By "average" is meant over space and time, since *both* of these independent variables enter into a determination of the mean suspension properties.

As emphasized in I, a large number of equations have been proposed attempting to describe the relationship existing between the suspension viscosity and the volumetric solids concentration ϕ. To these may be added several thorough reviews of experimental results (Frisch & Simha 1956, Rutgers 1962, Thomas 1965 and Jeffery & Acrivos 1976).

Initially, attention will be focused in section 2 on the traditional situation where the suspending fluid is Newtonian and the particle Reynolds number sufficiently small to permit neglect of inertial effects. Pertinent Darcy-scale fields of interest, e.g. velocity, stress, etc. will be introduced and subsequently related to the spatial average of the local fields.

Methods which are now relatively standard (Happel & Brenner 1965) are used in section 3 to describe the instantaneous quasistatic properties of the suspension. This scheme employs various intrinsic, purely geometric resistance matrices, arising from the linear character of the overall problem. Since these methods impose no limitations upon the shapes of the particles that fall within their purview, the analysis may be applied to nonspherical particles. However, when the particles are spherical (more generally centrosymmetric), geometric symmetry considerations yield important simplifications in the forms of these resistance matrices.

Time-averaging procedures are detailed in section 4 for the two-dimensional, almost periodic flows whose basic kinematics were investigated at length in I. A distinction between ergodic and nonergodic shearing processes is introduced. This distinction, though classical in the "ergodic theory" study of complex dynamical systems (Arnold & Avez 1968), appears new within the context of suspension flows. The time-average value is shown to depend upon the specific shearing process when it is not ergodic. Conversely, in the ergodic case, the time-average value is independent of the explicit process. The time average is then calculated in terms of the Fourier components of the function under consideration. This procedure reveals the explicit dependence of the average upon the particular shearing process. By way of example, these concepts are applied to a Couette flow between two regularly grooved corrugated planes, when the depth of the grooves is small compared with the gap width between the two planes.

Concentrated sphere suspensions are studied in detail in section 5. Lubrication theory is used for calculating the stress tensor via a rigorous version of the heuristic calculations of Frankel & Acrivos (1967). Possible extensions of the averaging procedure are made in Section 6 for various classes of nonNewtonian fluids. Some remarks on the influence of inertia complete the discussion.

2. BASIC EQUATIONS AND FIELDS AT THE DARCY SCALE

2.1. *Description of the problem*

Consider a spatially periodic suspension of identical particles immersed in an incompressible Newtonian fluid. The lattice Λ along which the particles are repetitively arranged is characterized by the second-order tensor

$$\mathbf{L} = \mathbf{l}_1\mathbf{e}_1 + \mathbf{l}_2\mathbf{e}_2 + \mathbf{l}_3\mathbf{e}_3, \tag{2.1}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is an orthonormal basis of the space $\mathbb{R}^3$ and $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3$ are three independent lattice vectors, which form a basis for Λ.

The position of an arbitrary point permanently affixed to the particles is denoted by $\mathbf{R}_\mathbf{n}$:

$$\mathbf{R}_\mathbf{n} = n_1\mathbf{l}_1 + n_2\mathbf{l}_2 + n_3\mathbf{l}_3, \quad [2.2]$$

where $\{n_1, n_2, n_3\} \equiv \{\mathbf{n}\}$ are integers. For spherical particles, the particular point is conveniently chosen to lie at the sphere center 0.

Stokes equations of motion governing the flow of a Newtonian fluid are

$$\nabla p = 2\mu\nabla \cdot \mathbf{S} \quad [2.3a]$$

and

$$\nabla \cdot \mathbf{v} = 0, \quad [2.3b]$$

where p is the pressure, μ the viscosity, $\mathbf{v}$ the velocity and

$$\mathbf{S} = \tfrac{1}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger\right] \quad [2.4]$$

the rate-of-strain tensor. Alternatively, in terms of the local stress tensor $\mathbf{P}$, defined by

$$\mathbf{P} = -p\mathbf{I} + 2\mu\mathbf{S}, \quad [2.5a]$$

equaton [2.3a] is equivalent to

$$\nabla \cdot \mathbf{P} = \mathbf{0}. \quad [2.5b]$$

At the surface $s_p\{\mathbf{n}\}$ of the particle positioned at $\mathbf{R}_\mathbf{n}$, adherence of the fluid requires satisfaction of the boundary condition

$$\mathbf{v}(\mathbf{R}) = \mathbf{U}_\mathbf{n} + \mathbf{\Omega}_\mathbf{n} \times \mathbf{r}, \quad [2.6]$$

with $\mathbf{U}_\mathbf{n}$ and $\mathbf{\Omega}_\mathbf{n}$ the particle's translational and angular velocities. Local position vector $\mathbf{r}$, defined by

$$\mathbf{R} = \mathbf{R}_\mathbf{n} + \mathbf{r}, \quad [2.7]$$

terminates on the particle surface (see figure 1).

Imagine the suspension subjected to a macroscopic, homogeneous, linear shear flow, characterized by the second-order tensor $\mathbf{G}$. The resulting local velocity field created at a point $\mathbf{R}$ of the spatially periodic suspension will be assumed to satisfy the "jump" condition

$$\mathbf{v}(\mathbf{R} + \mathbf{R}_\mathbf{n}) - \mathbf{v}(\mathbf{R}) = \mathbf{R}_\mathbf{n} \cdot \mathbf{G}, \quad [2.8]$$

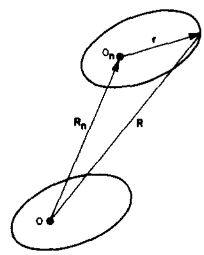

Figure 1. Geometry of the spatially periodic suspension. O and $O_\mathbf{n}$ are analogous points inside the particles.

where $\mathbf{G}$ is the position- and time-independent macroscopic velocity gradient. This relation is assumed to apply whether $\mathbf{R}$ lies inside or outside of a particle. Fluid and particle incompressibility require that $\mathbf{G}$ be traceless. Explicitly,

$$\operatorname{tr} \mathbf{G} = 0. \qquad [2.9]$$

This completes the general description of the problem. However, a nonspherical particle necessitates orientational considerations too. Characterization of this orientation can be achieved via a trio of body-fixed unit vectors permanently locked into, or otherwise affixed to, the particles, so as to rotate with them under the influence of the shear or other orienting torque (Brenner 1981). A precise account of the external forces and torques which act upon the particles will be rendered later.

2.2. *Darcy-scale kinematics*

Condition [2.8] is equivalent to assuming the gradient of the local velocity field to be spatially periodic:

$$\nabla \mathbf{v}(\mathbf{R}) = \nabla \mathbf{v}(\mathbf{R} + \mathbf{R}_n). \qquad [2.10]$$

According to a fundamental decomposition theorem (Brenner & Adler 1985), any tensor-valued field $\mathbf{v}$ whose gradient $\nabla \mathbf{v}$ is spatially periodic can be expressed as the sum

$$\mathbf{v}(\mathbf{R}) = \mathbf{R} \cdot \mathbf{G} + \tilde{\mathbf{v}}'(\mathbf{R}) \qquad [2.11]$$

of a spatially periodic field $\tilde{\mathbf{v}}'(\mathbf{R})$ and a linearly varying field $\mathbf{R} \cdot \mathbf{G}$, wherein

$$\mathbf{G} = \frac{1}{\tau_o} \oint_{\partial \tau_o} \mathrm{d}\mathbf{s}\, \mathbf{v} = \text{lattice constant} \qquad [2.12]$$

is a tensorial lattice constant whose rank exceeds that of $\mathbf{v}$ by one. Apart from possessing the same value throughout the entire medium, a lattice constant (Brenner & Adler 1985) possesses the property of being explicitly and implicitly independent of the mode of partitioning the lattice into unit cells; that is, it is an intrinsic property of the lattice, rather than of the unit cell, despite the fact that its value is derived by performing an integration over a unit cell.

The translational velocity $\mathbf{U}_n$ of particle $\{\mathbf{n}\}$ may similarly be decomposed as

$$\mathbf{U}_n = \mathbf{U}_0 + \mathbf{R}_n \cdot \mathbf{G}, \qquad [2.13]$$

in which $\mathbf{U}_0$ is the velocity at the locator point $\mathbf{R}_0$ of the particle in cell $\{\mathbf{0}\}$.

As a consequence of [2.6], [2.8] and [2.13] the angular velocity $\boldsymbol{\Omega}_n$ of particle $\mathbf{n}$ proves to be independent of $\mathbf{n}$. Accordingly, $\boldsymbol{\Omega}_n$ will henceforth be denoted unambiguously as $\boldsymbol{\Omega}$.

For future reference it is convenient to summarize here some basic integral identities. Over any closed surface S bounding a volume τ,

$$\oint_S \mathrm{d}\mathbf{s} = \mathbf{0}, \qquad [2.14a]$$

$$\oint_S \mathrm{d}\mathbf{s}\, \mathbf{R} = \mathbf{I}\tau, \qquad [2.14b]$$

and

$$\frac{1}{\tau} \oint_S \mathbf{R}\, \mathrm{d}\mathbf{s}\, \mathbf{R} = \mathbf{I}\overline{\mathbf{R}} + \overline{\mathbf{R}}\mathbf{I}, \qquad [2.14c]$$

in which

$$\bar{\mathbf{R}} = \frac{1}{\tau}\int_{\tau} \mathbf{R}\, d^3\mathbf{R} \qquad [2.15]$$

denotes the center of volume of τ. In the above, $\mathbf{I}$ is the dyadic idemfactor, $d\mathbf{s}$ an element of surface area having the direction of the outer normal to τ, and $d^3\mathbf{R}$ is a volume element.

It is necessary in the subsequent analysis to introduce an intermediate length scale $\mathcal{L}$ (Brenner & Adler 1985) such that

$$L \gg \mathcal{L} \gg l, \qquad [2.16]$$

where L and l are respectively characteristic linear dimensions of the suspension and unit cell. The various macroscopic, suspension- or Darcy-scale fields of physical interest are obtained by volume averaging of the relevant local fields over an intermediate volume $\mathcal{V} = O(\mathcal{L}^3)$. This volume is centered about the origin of the $\mathbf{R}$ coordinate system, defined by

$$\frac{1}{\mathcal{V}}\int_{\mathcal{V}} \mathbf{R}\, d^3\mathbf{R} = \mathbf{0}. \qquad [2.17]$$

An elementary application of this averaging procedure yields the mean, or Darcy-scale, velocity vector

$$\bar{\mathbf{v}} = \frac{1}{\mathcal{V}}\int_{\mathcal{V}} \mathbf{v}\, d^3\mathbf{R}. \qquad [2.18]$$

Though this field elicits no real interest in and of itself, it nevertheless serves to introduce several important subsequent relations and Darcy-scale dependent variables.

Among these is the mean interstitial fluid velocity vector $\bar{\mathbf{v}}^*$, defined as

$$\bar{\mathbf{v}}^* = \frac{1}{\tau_f}\int_{\tau_f\{\mathbf{n}\}} \check{\mathbf{v}}'\, d^3\mathbf{R}, \qquad [2.19]$$

with $\tau_f\{\mathbf{n}\}$ the interstitial fluid volume within cell $\{\mathbf{n}\}$. This definition suggests the further decomposition

$$\check{\mathbf{v}}' = \bar{\mathbf{v}}^* + \check{\mathbf{v}}(\mathbf{R}) \qquad [2.20]$$

of the spatially periodic component of the velocity field appearing in [2.11] into a nonzero mean part $\bar{\mathbf{v}}^*$ and a zero mean part $\check{\mathbf{v}}$, satisfying the relation

$$\frac{1}{\tau_f}\int_{\tau_f} \check{\mathbf{v}}(\mathbf{R})\, d^3\mathbf{R} = \mathbf{0}. \qquad [2.21]$$

Thus, the penultimate decomposition (cf. [2.26]) of the velocity field may be written as

$$\mathbf{v}(\mathbf{R}) = \bar{\mathbf{v}}^* + \mathbf{R}\cdot\mathbf{G} + \check{\mathbf{v}}(\mathbf{R}), \qquad [2.22]$$

whose fundamental significance will appear later.

The macroscopic velocity gradient $\mathbf{G}$ represents the average value (over the entire cell τ_o) of the *microscopic* gradient $\nabla\mathbf{v}$, when the velocity field is continuous across the fluid-particle interface. This is easily proved by observing that, by definition,

$$\overline{\nabla \mathbf{v}} = \frac{1}{\tau_o}\int_{\tau_o} \nabla \mathbf{v}\, d^3\mathbf{R} = \frac{1}{\tau_o}\oint_{\partial\tau_o} d\mathbf{s}\, \breve{\mathbf{v}}' + \mathbf{G}, \qquad [2.23]$$

upon employing the decomposition [2.11]. The first integral vanishes as a consequence of the general fact (Brenner & Adler 1985) that the flux of any spatially periodic field $\breve{\mathbf{h}}$ across the closed surface of a cell of any shape is identically zero.

As usual, the macroscopic gradient $\mathbf{G}$ may be decomposed into symmetric and antisymmetric parts, denoted by $\overline{\mathbf{S}}$ and $\overline{\mathbf{\Lambda}}$, respectively:

$$\mathbf{G} = \overline{\mathbf{S}} + \overline{\mathbf{\Lambda}}. \qquad [2.24]$$

To the antisymmetric tensor $\overline{\mathbf{\Lambda}} = \frac{1}{2}[\overline{\nabla \mathbf{v}} - (\overline{\nabla \mathbf{v}})^\dagger]$, one can associate the pseudovector

$$\overline{\omega} = \tfrac{1}{2}\overline{\nabla \times \mathbf{v}}, \qquad [2.25]$$

representing the macroscopic angular velocity of the suspension. Consequently, [2.2] can be further decomposed as

$$\mathbf{v}(\mathbf{R}) = \bar{\mathbf{v}}^* + \overline{\omega} \times \mathbf{R} + \mathbf{R} \cdot \overline{\mathbf{S}} + \breve{\mathbf{v}}(\mathbf{R}). \qquad [2.26]$$

Introduce [2.13] and [2.26] into boundary condition [2.6] to obtain

$$\breve{\mathbf{v}} = (\mathbf{U}_0 - \bar{\mathbf{v}}^*) + (\mathbf{\Omega} - \overline{\omega}) \times \mathbf{r} - \mathbf{r} \cdot \overline{\mathbf{S}} \quad \text{on } s_p. \qquad [2.27]$$

This linear form will prove fundamental in the subsequent analysis.

2.3. *Darcy-scale dynamics*

We shall here extend to sheared suspensions, macroscopic Darcy-scale entitities originally introduced (Brenner & Adler 1985) for flow in *stationary porous media.* Included among these are external body-force and couple densities, and stresses. The explicit formulation chosen will be of sufficient generality to permit the resulting equations and properties to be applied to situations where the suspending fluid manifests several types of non-Newtonian behavior.

The basic properties to be discussed arise as a direct consequence of the assumption [2.10] of a spatially periodic velocity gradient. As an immediate consequence of its definition [2.4], the local rate of strain $\mathbf{S}$ is spatially periodic. Stokes equations [2.3a] then show that the pressure gradient too is spatially periodic. Explicitly,

$$\nabla \mathbf{v},\ \mathbf{S},\ \nabla p \text{ are spatially periodic.} \qquad [2.28]$$

Hence, we can apply to the local pressure field p the fundamental decomposition theorem [2.11] already used for $\mathbf{v}$. This yields

$$p(\mathbf{R}) = \breve{p}(\mathbf{R}) - \mathbf{R} \cdot \overline{\mathbf{F}}, \qquad [2.29]$$

where $\breve{p}(\mathbf{R})$ is a spatially periodic scalar field and $\overline{\mathbf{F}}$ a vectorial lattice constant. As in the generic relation [2.12], $\overline{\mathbf{F}}$ is the vector

$$\overline{\mathbf{F}} = -\frac{1}{\tau_o}\oint_{\partial\tau_o} d\mathbf{s}\, p = \text{a lattice constant.} \qquad [2.30]$$

A relation exists between $\overline{\mathbf{F}}$ and the hydrodynamic force

$$\mathbf{F} = \int_{s_p\{\mathbf{n}\}} \mathbf{ds} \cdot \mathbf{P} \qquad [2.31]$$

exerted by the fluid on the particle in cell $\{\mathbf{n}\}$. Here, $\mathbf{ds}$ is directed out of the particle. Stokes' equation [2.5b] allows performance of the stress integration over the cell surface, rather than the particle surface; that is,

$$\mathbf{F} = \oint_{\partial\tau_o\{\mathbf{n}\}} \mathbf{ds} \cdot \mathbf{P}. \qquad [2.32]$$

The stress $\mathbf{P}$ can be decomposed in accordance with [2.5a]. As a consequence of [2.28] the contribution of the periodic component over the cell surface vanishes, whereupon [2.32] reduces to

$$\mathbf{F} = -\oint_{\partial\tau_o\{\mathbf{n}\}} \mathbf{ds}\, p. \qquad [2.33]$$

Comparison with [2.30] reveals that

$$\mathbf{F} = \tau_o\overline{\mathbf{F}}. \qquad [2.34]$$

Since both τ_o and $\overline{\mathbf{F}}$ are lattice constants it follows that

$$\mathbf{F} = \text{a lattice constant}. \qquad [2.35]$$

Let $\mathbf{F}^{(e)}$ denote the total external force exerted upon the particle(s) in cell $\{\mathbf{n}\}$. This force may arise from gravity, for instance, in the case of nonneutrally buoyant particles. As inertia is neglected throughout the present work, Newton's laws of motion require that the sum of the various forces exerted upon the particle(s) be zero. Explicitly,

$$\mathbf{F} + \mathbf{F}^{(e)} = \mathbf{0}. \qquad [2.36]$$

Thus, according to [2.35], the external force exerted on each particle is a lattice constant. From a physical point of view no other possibility exists.

As a consequence of [2.8], [2.34] and [2.36], the pressure field is spatially periodic when the net external force exerted on each of the particles is identically zero, such as obtains for a suspension of neutrally buoyant particles.

The hydrodynamic couple exerted by the fluid upon the entire contents (fluid plus particles) of $\mathcal{V}$ is defined as

$$\mathbf{N} = \frac{1}{\mathcal{N}} \oint_{\partial\mathcal{V}} \mathbf{R} \times (\mathbf{dS} \cdot \mathbf{P}), \qquad [2.37]$$

where $\mathcal{N}$ is the number of unit cells contained in the intermediate volume $\mathcal{V}$, and $\partial\mathcal{V}$ is the external surface. In view of [2.29] the stress tensor $\mathbf{P}$ may be decomposed into periodic and aperiodic components:

$$\mathbf{P} = \breve{\mathbf{P}} + \mathbf{I}(\mathbf{R} \cdot \overline{\mathbf{F}}), \qquad [2.38]$$

wherein

$$\breve{\mathbf{P}} = -\mathbf{I}\breve{p} + 2\mu\mathbf{S}. \qquad [2.39]$$

In [2.37] the linearly varying portion of P can be eliminated since the origin of $\mathbf{R}$ coincides with the center of gravity of the volume $\mathcal{V}$. The remaining integration reduces to a quadrature over the boundary $\partial\tau_o$ of a single unit cell. Consequently,

$$\mathbf{N} = \oint_{\partial\tau_o} \mathbf{r} \times (\mathbf{ds} \cdot \breve{\mathsf{P}}) + O(l). \qquad [2.40a]$$

Equivalently,

$$\mathbf{N} = \oint_{\partial\tau_o} \mathbf{r} \times (\mathbf{ds} \cdot \mathsf{P}) + O(l). \qquad [2.40b]$$

By definition, the external couple $\mathbf{N}^{(e)}$ exerted upon the particles by an agency lying outside of the suspension is

$$\mathbf{N}^{(e)} = \frac{1}{\mathcal{N}} \sum_{\mathbf{n}\in\mathcal{V}} \int_{s_p\{\mathbf{n}\}} \mathbf{R} \times (\mathbf{ds}' \cdot \mathsf{P}), \qquad [2.41]$$

where $\mathbf{ds}'$ is directed into the particle surface(s) s_p. Since Stokes' equation [2.5b] may be equivalently written as

$$\nabla \cdot (\mathsf{P} \times \mathbf{R}) = \mathbf{0}, \qquad [2.42]$$

the fundamental equilibrium equation

$$\mathbf{N} + \mathbf{N}^{(e)} = \mathbf{0} \qquad [2.43]$$

relating the hydrodynamic and external couples is easily derived. A more convenient expression for the external couple [2.41] is

$$\mathbf{N}^{(e)} = \int_{s_p} \mathbf{r} \times (\mathbf{ds}' \cdot \mathsf{P}) + \frac{1}{\tau_o}\left(\int_{\tau_o} \mathbf{r}\, d^3\mathbf{R}\right) \times \mathbf{F} + O(l). \qquad [2.44]$$

It can be proved (Brenner & Adler 1985) that $\mathbf{N}$ and $\mathbf{N}^{(e)}$ actually qualify as couples, since they do not depend upon the choice of origin from which the local position vector $\mathbf{r} = \mathbf{R}_\mathbf{n} - \mathbf{R}$ is to be measured.

2.4. *Macroscopic equations*

Next, consider the Darcy-scale stress, defined as the volume average,

$$\bar{\mathsf{P}} = \frac{1}{\mathcal{V}} \int_{\mathcal{V}} \mathsf{P}\, d^3\mathbf{R}, \qquad [2.45]$$

of the local stress P. Since $\mathcal{V}$ is centered about $\mathbf{R} = \mathbf{0}$, the preceding reduces to

$$\bar{\mathsf{P}} = \frac{1}{\tau_o} \int_{\tau_o} \breve{\mathsf{P}}\, d^3\mathbf{R}, \qquad [2.46]$$

upon using [2.17] and [2.38]. (Of course the same result holds independently of the choice of origin for a neutrally buoyant suspension, $\bar{\mathbf{F}}^{(e)} = \mathbf{0}$.)

According to [2.5b], P is divergence free. As such, it may be expressed in the form

$$\mathsf{P} = \nabla \cdot (\mathsf{P}\mathbf{R}). \qquad [2.47]$$

Consequently, the Darcy-scale stress [2.45] may be written equivalently as

$$\bar{\mathbf{P}} = \frac{1}{\mathcal{V}} \oint_{\partial\mathcal{V}} \mathbf{R}\, \mathrm{d}\mathbf{s} \cdot \mathbf{P} \quad [2.48a]$$

or

$$\bar{\mathbf{P}} = \frac{1}{\tau_o} \oint_{\partial\tau_o} \mathbf{r}\, \mathrm{d}\mathbf{s} \cdot \breve{\mathbf{P}}, \quad [2.48b]$$

upon employing [2.38], [2.15], [2.14c], [2.17] and [2.7].

As usual, $\bar{\mathbf{P}}$ may be decomposed into its isotropic and deviatoric portions

$$\bar{\mathbf{P}} = -\mathbf{I}\bar{p} + \bar{\mathbf{T}}, \quad [2.49]$$

in which

$$\bar{p} = -\tfrac{1}{3}\mathbf{I} : \bar{\mathbf{P}} \quad [2.50a]$$

and

$$\bar{\mathbf{T}} = \frac{1}{\tau_o} \oint_{\partial\tau_o} \left(\mathbf{r}\mathbf{I} - \frac{1}{3}\mathbf{I}\mathbf{r}\right) \cdot \mathrm{d}\mathbf{s} \cdot \mathbf{P}. \quad [2.50b]$$

This last equality holds only to within an error of $O(l)$, which is negligible.

The deviatoric part $\bar{\mathbf{T}}$ of the macroscopic stress tensor may itself be decomposed into symmetric and antisymmetric portions. The symmetric part, denoted by $\bar{\tau}$, adopts the form

$$\bar{\tau} = \tfrac{1}{2}(\bar{\mathbf{P}} + \bar{\mathbf{P}}\dagger) - \tfrac{1}{3}\mathbf{I}(\mathbf{I} : \bar{\mathbf{P}}), \quad [2.51]$$

wherein the dagger represents the transposition operator. This expression is more easily transformed by returning to the definition [2.45] of the macroscopic stress. Integrate over $\mathcal{V}_f$ and $\mathcal{V}_p$, and use the divergence theorem to express the integrals over $\mathcal{V}_p$ as equivalent surface integrals over the particle surfaces contained in $\mathcal{V}$. Conventional manipulations permit further reductions of the resulting integrals to comparable integrals extended over the unit cell. This eventually yields

$$\bar{\tau} = \frac{1}{2\tau_o} \int_{s_p} \left(\mathrm{d}\mathbf{s} \cdot \breve{\mathbf{P}}\mathbf{r} + \mathbf{r}\, \mathrm{d}\mathbf{s} \cdot \breve{\mathbf{P}} - \frac{2}{3}\mathbf{I}\, \mathrm{d}\mathbf{s} \cdot \breve{\mathbf{P}} \cdot \mathbf{r}\right) + \frac{1}{2\tau_o} \int_{\tau_f} \left[\breve{\mathbf{P}} + \breve{\mathbf{P}}\dagger - \frac{2}{3}\mathbf{I}(\mathbf{I} : \breve{\mathbf{P}})\right] \mathrm{d}^3\mathbf{R}. \quad [2.52]$$

This second integral may be equivalently expressed as (cf. [2.5a])

$$\frac{2\mu}{\tau_o} \int_{\tau_f} \mathbf{S}\, \mathrm{d}^3\mathbf{R}. \quad [2.53]$$

Further modifications result from utilizing eq [2.4], the divergence theorem and [2.12] to obtain the identity

$$\frac{2\mu}{\tau_o} \int_{\tau_f} \mathbf{S}\, \mathrm{d}^3\mathbf{R} = 2\mu \bar{\mathbf{S}} - \frac{\mu}{\tau_o} \int_{s_p} (\mathrm{d}\mathbf{s}\, \mathbf{v} + \mathbf{v}\, \mathrm{d}\mathbf{s}). \quad [2.54]$$

The last surface integral vanishes as a consequence of the boundary condition [2.6] and the general identities [2.14].

Together, the preceding relations combine to yield the formula

$$\bar{\tau} = 2\mu\,\bar{\mathbf{S}} + \frac{1}{\tau_o}\mathbf{A}, \qquad [2.55]$$

where

$$\mathbf{A} = \int_{s_p}\left[\frac{1}{2}(\mathrm{d}\mathbf{s}\cdot\breve{\mathbf{P}}\cdot\mathbf{r} + \mathbf{r}\cdot\mathrm{d}\mathbf{s}\cdot\breve{\mathbf{P}}) - \frac{1}{3}\mathbf{I}(\mathrm{d}\mathbf{s}\cdot\breve{\mathbf{P}}\cdot\mathbf{r})\right] \qquad [2.56]$$

is the particle stress, representing the contribution of the particles to the macroscopic stress. In the preceding, the surface element $\mathrm{d}\mathbf{s}$ is directed out of the particle into the fluid.

The antisymmetric part of the macroscopic stress tensor $\bar{\mathbf{P}}$ may be represented alternatively by a pseudovector upon invoking the identity

$$\bar{\mathbf{P}} - \bar{\mathbf{P}}\dagger = \boldsymbol{\epsilon}\cdot\bar{\mathbf{P}}_x, \qquad [2.57]$$

with $\boldsymbol{\epsilon}$ the unit alternating triadic and $\bar{\mathbf{P}}_\times$ the vector invariant of $\bar{\mathbf{P}}$. From [2.48b],

$$\bar{\mathbf{P}}_\times = \frac{1}{\tau_o}\oint_{\partial\tau_o}\mathbf{r}\times(\mathrm{d}\mathbf{s}\cdot\breve{\mathbf{P}}). \qquad [2.58]$$

Comparison of [2.58] with [2.40a] yields the relation

$$\bar{\mathbf{P}}_\times = \mathbf{N}/\tau_o. \qquad [2.59]$$

Equivalently, as a consequence of [2.43],

$$\bar{\mathbf{P}}_\times + \bar{\mathbf{N}}^{(e)} = \mathbf{0}, \qquad [2.60]$$

in which

$$\bar{\mathbf{N}}^{(e)} = \mathbf{N}^{(e)}/\tau_o \qquad [2.61]$$

is the external body-couple volumetric density. Equation [2.60] represents a degenerate angular momentum equation for a polar continuum (Dahler & Scriven 1963, Brenner 1970 and Brenner & Weissman 1972), from which couple stress and intrinsic angular momentum effects are absent.

2.5. *Energy equation*

The local rate Φ of mechanical energy dissipation (per unit time per unit volume) at each point $\mathbf{R}$ within the interstitial fluid is classically given by (Aris 1962)

$$\Phi = \mathbf{T}\dagger : \nabla\mathbf{v}, \qquad [2.62]$$

where $\mathbf{T} = \mathbf{P} + \mathbf{I}p$ is the deviatoric part of the local stress $\mathbf{P}$. Thermodynamic principles pertaining to irreversible entropy production (Landau & Lifshitz 1958) show that Φ is a non-negative quantity:

$$\Phi \geq 0. \qquad [2.63]$$

Equality, in the context of the above inequality, holds only for a rigid body motion.

The time rate of dissipation $\overline{\Phi}$ (per unit of superficial volume) within the intermediate volume $\mathcal{V}$ is defined as

$$\overline{\Phi} = \frac{1}{\mathcal{V}} \int_{\mathcal{V}_f} \Phi \, d^3\mathbf{R}. \qquad [2.64]$$

It too is obviously a non-negative quantity. This macroscopic dissipation rate can be functionally expressed in terms of the various macroscopic dynamic and kinematic parameters occurring in the Darcy-scale characterization of the present problem by use of the alternative form,

$$\Phi = \nabla \cdot (\mathbf{P} \cdot \mathbf{v}), \qquad [2.65]$$

of [2.62] in the integrand of [2.64] (cf. [2.5b]).

In conjunction with the divergence theorem and the decompositions [2.38] and [2.11] for $\mathbf{P}$ and $\mathbf{v}$, this eventually yields

$$\overline{\Phi} = (\tau_f/\tau_o)\overline{\mathbf{F}}^e \cdot (\mathbf{U}_0 - \overline{\mathbf{v}}^*) + \overline{\mathbf{N}}^{(e)} \cdot (\boldsymbol{\Omega} - \overline{\boldsymbol{\omega}}) + \overline{\boldsymbol{\tau}} : \overline{\mathbf{S}}, \qquad [2.66]$$

to within an error of $O(l)$.

The preceding expression serves many useful purposes. First, it shows clearly that the velocity of physical interest is the relative interstitial velocity $\overline{\mathbf{v}}^* - \mathbf{U}_0$; hence the utility of decomposition [2.22] is justified *a posteriori*. Second, [2.66] can be used to prove uniqueness of the solution under prescribed macroscopic conditions, as outlined in the next subsection. Last, the symmetry of important matrices (cf. [3.4]) can be readily inferred from [2.66] by an application of the Lorentz reciprocal theorem (Hinch 1972), as can their nonnegativity too, as a consequence of the inequality $\overline{\Phi} \geq 0$.

Only the bare outlines of a uniqueness proof for the microscopic problem will be furnished here. The solution $(\mathbf{v}, p)$ of the system of equations [2.3] subject to boundary conditions [2.6] is unique at each point $\mathbf{R}$ (modulo a rigid body motion), provided that the values of the following three parameters are prescribed:

$$\begin{aligned} &\text{(i)} \quad \overline{\mathbf{v}}^* - \mathbf{U}_0 \quad \text{or} \quad \overline{\mathbf{F}}^{(e)}, \\ &\text{(ii)} \quad \overline{\boldsymbol{\omega}} - \boldsymbol{\Omega} \quad \text{or} \quad \overline{\mathbf{N}}^{(e)}, \\ &\text{(iii)} \quad \overline{\mathbf{S}} \quad \text{or} \quad \overline{\boldsymbol{\tau}}. \end{aligned} \qquad [2.67]$$

To prove uniqueness under the prescribed data, assume the existence of at least two different solutions $(\mathbf{v}', p')$ and $(\mathbf{v}'', p'')$ of the specified system of equations and conditions. Then, according to [2.66], it is easy to show that the macroscopic and, hence, microscopic dissipation rates associated with the difference between these two solutions is zero. An alternate expression for the local rate [2.62] is given by the quadratic form

$$\Phi = 2\mu \mathbf{S} : \mathbf{S}. \qquad [2.68]$$

Hence, the rate-of-strain tensor $\mathbf{S}' - \mathbf{S}''$ associated with the difference fields vanishes everywhere. This implies (Lamb 1932) that the difference velocity field $\mathbf{v}' - \mathbf{v}''$ between the two possible solutions is a rigid body motion. Furthermore, Stokes equations then show the difference pressure $p' - p''$ to be zero (modulo an arbitrary constant, which is physically irrelevant for incompressible fluids).

3. INSTANTANEOUS PROPERTIES

With the preliminaries developed in section 2, it is a relatively simple matter to apply classical matrix methods (Happel & Brenner 1965) to the analysis of these linear systems. It is interesting to note here that these methods have already been applied to multiparticle systems (Brenner & O'Neill 1972). However, the formidable relative trajectory problems which ensue are tractable only for two-particle systems, though within this limited two-particle context the scheme is very potent (Adler 1981). The spatial periodicity assumption introduced here reduces the multiparticle problem to a single particle problem, since only the contents of one cell need be considered in the analysis. This restores the tractability of the general trajectory matrix scheme.

For completeness, a relative interstitial velocity ($\bar{\mathbf{v}}^* - \mathbf{U}_0$) contribution is included. The existence of such "slip" velocities generally arises from the existence of external forces—as in the case of flow through porous media (Brenner & Adler 1985), though such external forces are absent for *neutrally buoyant* suspensions. Nevertheless, slip velocities can arise even for neutrally buoyant particles if they lack a center of symmetry, which fact accounts for their inclusion here, at least in part. Inasmuch as the linear scheme which ensues is rather classical in format and scope, its exposition will be greatly abbreviated.

Following Brenner & O'Neill (1972), it is more convenient to use the spatially periodic field $\check{\mathbf{v}}(\mathbf{R})$ introduced in [2.22] than the actual velocity field $\mathbf{v}(\mathbf{R})$. In terms of $\check{\mathbf{v}}(\mathbf{R})$, Stokes equations of motion may be expressed as

$$\nabla p = \mu \nabla \cdot \nabla \check{\mathbf{v}} \tag{3.1a}$$

and

$$\nabla \cdot \check{\mathbf{v}} = 0, \tag{3.1b}$$

to which are adjoined the supplementary requirements:

$$\check{\mathbf{v}} = (\mathbf{U}_0 - \bar{\mathbf{v}}^*) + (\boldsymbol{\Omega} - \bar{\boldsymbol{\omega}}) \times \mathbf{r} - \mathbf{r} \cdot \bar{\mathbf{S}} \quad \text{on } s_p, \tag{3.2}$$

$$\int_{\tau_f} \check{\mathbf{v}} \, d^3\mathbf{R} = \mathbf{0} \tag{3.3a}$$

and

$$\check{\mathbf{v}} = \text{spatially periodic}, \tag{3.3b}$$

which follow from [2.27], [2.21] and [2.11].

Linearity of the above system requires that each macroscopic dynamical quantity be linear in the kinematical entities $\mathbf{U}_0 - \bar{\mathbf{v}}^*$, $\boldsymbol{\Omega} - \bar{\boldsymbol{\omega}}$ and $\bar{\mathbf{S}}$. In particular, the hydrodynamic force $\mathbf{F}$, couple $\mathbf{N}$ and stresslet A exerted by the fluid on a particle may be expressed in terms of the "grand resistance matrix" as

$$\begin{pmatrix} \mathbf{F} \\ \mathbf{N} \\ \mathsf{A} \end{pmatrix} = \mu \begin{pmatrix} {}^t\mathbf{K} & \mathbf{C} & \boldsymbol{\phi} \\ (\tau_f/\tau_o)\mathbf{C}^\dagger & {}^r\mathbf{K} & \boldsymbol{\tau} \\ (\tau_f/\tau_o)\mathsf{M} & \mathsf{N}^* & \mathsf{Q} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{v}}^* - \mathbf{U}_0 \\ \bar{\boldsymbol{\omega}} - \boldsymbol{\Omega} \\ \bar{\mathbf{S}} \end{pmatrix}, \tag{3.4}$$

subject to the following symmetry relations:

$$\begin{gathered} {}^tK_{ji} = {}^tK_{ij}, \quad {}^rK_{ji} = {}^rK_{ij}, \\ N^*_{ijk} = \tau_{kij}, \quad M_{ijk} = \phi_{kij}, \quad Q_{ijkl} = Q_{klij}. \end{gathered} \tag{3.5}$$

These kinetic symmetry relations are a consequence of the Lorentz reciprocal theorem, applied to the fluid volume τ_f contained within the unit cell. The extraneous factor τ_f/τ_o appearing in (3.4) arises from the fact that our definition [2.31] of the force includes the average pressure gradient (cf. Brenner & Adler 1985 for a further discussion of this point in the porous medium context, when the force is deduced from the "homogenization" method of Bensoussan *et al.* 1978).

In consequence of the symmetry $\bar{S}_{ij} = \bar{S}_{ji}$ of $\bar{\mathbf{S}}$, one can arbitrarily set

$$\phi_{ijk} = \phi_{ikj}, \quad \tau_{ijk} = \tau_{ikj}, \quad Q_{ijkl} = Q_{ijlk} \tag{3.6}$$

in the preceding, without loss of generality. Further reduction in the number and nature of the independent components of these matrices occurs as a result of the incompressibility condition $\mathbf{I} : \bar{\mathbf{S}} = 0$ (i.e. $\bar{S}_{ii} = 0$). Moreover, according to the definition of the particle stress A_{ij}, which is symmetric and traceless, it necessarily follows that

$$M_{iik} = 0, \quad N_{iik} = 0, \quad Q_{iikl}\bar{S}_{kl} = 0. \tag{3.7}$$

These relationships, together with an obvious classification of the phenomenological coefficient matrices into pseudo (axial) and true (polar) tensors, greatly reduce the number of independent, nonzero scalar coefficients which must be calculated in the general case.

The grand resistance matrix defined in [3.4] is a function only of the instantaneous geometrical configuration of the system. This consists of the fixed particle shapes and the variable relative particle positions and orientations. As such, geometrical symmetry arguments further reduce the number of independent nonzero components of the previous tensors for particular choices of coordinate systems (e.g. "principal axis" systems). The development itself and supporting arguments are basically the same as for a single particle in an unbounded fluid (Happel & Brenner 1965), though the point group symmetry elements of the lattice must be considered simultaneously (Brenner & Adler 1985) in a crystallographic sense. Thus, it will suffice to restrict ourselves to a few salient comments.

Our immediate goals pertain specifically to the important case of centrosymmetric particles, including spheres, ellipsoids, disks, rods, etc. (cf. Brenner 1974 for a complete account). Note that a suspension composed of such particles is itself centrosymmetric, since the lattice is always centrally symmetric. In such circumstances, geometric symmetry arguments (Brenner 1974) reduce the matrix [3.4] to the form

$$\begin{pmatrix} \mathbf{F} \\ \mathbf{N} \\ \mathbf{A} \end{pmatrix} = \mu \begin{pmatrix} {}^t\mathbf{K} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & {}^r\mathbf{K} & \boldsymbol{\tau} \\ \mathbf{0} & \mathbf{N}^* & \mathbf{Q} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{v}}^* - \mathbf{U}_0 \\ \bar{\boldsymbol{\omega}} - \boldsymbol{\Omega} \\ \bar{\mathbf{S}} \end{pmatrix}. \tag{3.8}$$

This form significantly simplifies the algebraic structure of the problem. In particular, no coupling now exists between the translational motion and the other two motions. This suggests the possibility of considering the translational motion independently of the angular and shearing motions.

There is, however, an indirect type of coupling between translational and shear motions, whose existence is not apparent from previous considerations, but which is of interest in at least several cases. As already mentioned, the grand resistance matrix depends upon the geometry of the system. A shearing motion will modify this geometry (see I) and with it the translational resistance dyadic ${}^t\mathbf{K}$ too. Hence, the instantaneous sedimentation velocity of such a suspension when subjected to steady external forces, e.g. gravity, will be time dependent. This situation will be addressed in section 4.4 (cf. [4.32a]).

4. AVERAGING PROCEDURE FOR A SUSPENSION OF IDENTICAL SPHERES IN A TWO-DIMENSIONAL SHEAR FLOW

We shall here calculate the average properties of a suspension for the only situation whose kinematical properties are currently completely known. This corresponds to a suspension composed of identical spheres undergoing one of the two-dimensional motions studied in I. Attention will be further focused upon a macroscopically simple shear flow, which represents the most important case encountered in practice. It will first be demonstrated that the instantaneous geometry of the suspension can be described by a single vector. Subsequently, we show how to calculate the average of any variable that is functionally dependent upon this vector. Results obtained in this manner are then applied, discussed and generalized.

4.1. *Geometry of a simple shear flow*

Recall the major features of a simple shear flow as described in I. Two cases were distinguished—slide flow and tube flow, respectively represented in figures 2 and 3. Velocity components are given by

$$u = Gz, \quad v = 0, \quad w = 0. \qquad [4.1]$$

The static or time-independent portion of the problem, namely those geometrical elements that remain unaltered by the motion, are the lattice vectors $\mathbf{l}_1$, $\mathbf{l}_2$, and l_{33}. On the other hand, the kinematic portion of the problem is described by the projection of $\mathbf{l}_3$ onto the x-y plane, which plane is assumed to contain the vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ [see figure 2(a)]. Actually, only the value of this projection modulo the lattice vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ need be known, as depicted in figure 2(b). To further simplify the representation, the inverse transformation L'^{-1}, can be applied to the two-dimensional lattice $\mathbf{l}_1$, $\mathbf{l}_2$ in order to obtain the integer lattice Y, defined in I. In this representation, we define

$$\mathbf{l}(t) \equiv \mathsf{L}'^{-1} \cdot \mathbf{l}'_3, \quad [\text{mod}(1, 1)], \qquad [4.2]$$

where $\mathbf{l}'_3$ is the projection of $\mathbf{l}_3$ onto the x-y plane.

This same reduction can be performed for the tube flow depicted in figure 3. Here, the static part of the problem is determined by both $\mathbf{l}_1$ and the projections of $\mathbf{l}_2$ and $\mathbf{l}_3$ onto the y-z plane, whereas the kinematic part is represented by the projections of $\mathbf{l}_2$ and $\mathbf{l}_3$ onto the x axis, namely

$$l_{21} = Gl_{23}t + \text{const}, \qquad [4.3a]$$

$$l_{31} = Gl_{33}t + \text{const}. \qquad [4.3b]$$

These two scalars may be regarded as the components of a single vector $\mathbf{l}'$:

$$\mathbf{l}' = (l_{21}, l_{31}). \qquad [4.4]$$

Since interest exists in these projections only mod $|\mathbf{l}_1|$, $\mathbf{l}'$ can be divided by $|\mathbf{l}_1|$ in order to represent events in the unit square, as before. Hence, $\mathbf{l}$ here denotes

$$\mathbf{l}(t) = \frac{1}{|\mathbf{l}_1|}(l_{21}, l_{31}), \quad [\text{mod}(1, 1)]. \qquad [4.5]$$

The two possible situations which may arise in a simple shear flow are therefore conveniently parametrized by a unique vector $\mathbf{l}(t)$, given either by [4.2] or [4.5]. Accordingly, in the following it will no longer be necessary to distinguish between the separate cases of slide and tube flows.

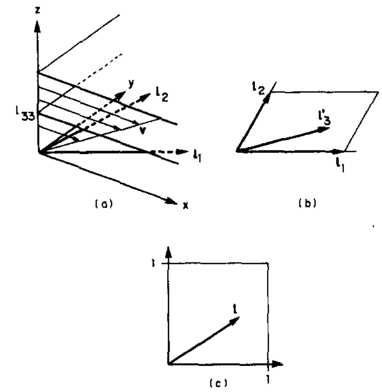

Figure 2. Slide flow. The planes are given schematically in (a). The projection $\mathbf{l}_3'$ of the basic vector $\mathbf{l}_3$ is shown in the parallelogram $\mathbf{l}_1$, $\mathbf{l}_2$ in (b). Application of equation [4.2] transforms this parallelogram into the unit square shown in (c).

4.2. *Ergodic properties and averaging*

Consider a function f which depends only on the geometry of the suspension—for example, in the case of spheres, upon the geometry of the lattice $\Lambda(t)$. The function f may be of any tensorial rank. For simple shear flow it will generally depend on both the static and kinematic components of the lattice. In order to simplify the representation, the static components will not explicitly appear among the arguments of f. Thus, notationally we shall write

$$f[\mathsf{L}(t)] \equiv f[\mathbf{l}(t)], \qquad [4.6]$$

with $\mathbf{l}(t)$ the vector previously defined. Of course, when actually performing specific calculations on a lattice, its static elements need be known too.

The quantity of physical interest will normally prove to be not the instantaneous value of the function f, but rather its time average, defined as

$$\langle f \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T f[\mathbf{l}(t)]\, \mathrm{d}t. \qquad [4.7]$$

In order to effect this integration, the behavior of $\mathbf{l}(t)$ within the unit square must be known. According to Weyl's theorem (cf. I) and its main consequences, a distinction must be drawn between two possible cases, depending upon the rational or the irrational properties of the vector $\mathbf{l}(t)$. More precisely, to within a shift in time scale (which is obviously irrelevant in

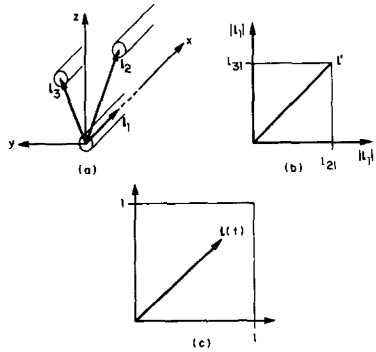

Figure 3. Tube flow. The tubes are depicted schematically in (a), together with the basic lattice vectors. The definition of $\mathbf{l}'$ is illustrated in (b). This square may be transformed into a unit square when equation [4.5] is applied, as shown in (c).

view of [4.7]), one may always write

$$l_x = t + l_{x0}, \quad l_y = \xi t, \tag{4.8}$$

in which t is now dimensionless. Here, l_{x0} denotes the value of l_x at time $t = 0$. Consequently, a distinction must now be drawn between rational and irrational values of the scalar ξ.

4.2.1. *ξ is rational.* Of course, the choice of time origin is without any import, whereupon $t = 0$ may be arbitrarily chosen such that $l_y = 0$. With this choice,

$$\begin{aligned} l_x &= t + l_{x0}, \quad (\text{mod } 1), \\ l_y &= (p/q)t, \quad (\text{mod } 1), \end{aligned} \tag{4.9}$$

where $\xi = p/q$, with p and q irreducible integers ($p \neq 0$). The trajectory of the vector $\mathbf{l}(t)$ on the unit square is illustrated in Fig. 4(a). Note that the pattern depends upon l_{x0}.

The period of the phenomenon is q. As such, the integral [4.7] may be expressed as

$$\langle f \rangle = \frac{1}{q} \int_0^q f[\mathbf{l}(t)]\, dt. \tag{4.10}$$

Denote by $\mathcal{L}$ the pattern made by $\mathbf{l}$ on the unit square, and by L its total length. The integral $\langle f \rangle$ is then modified as

$$\langle f \rangle = \frac{1}{L} \int_{\mathbf{l} \in \mathcal{L}} f(\mathbf{l})\, \mathrm{d}s, \tag{4.11}$$

whose computation now appears in the guise of a purely geometric, time-independent problem; ds is the differential element of length on the pattern $\mathcal{L}$. Observe that the pattern, and thus the average $\langle f \rangle$, depends upon the two parameters ξ and l_{x0}.

4.2.2. *ξ is irrational.* In this case $\mathbf{l}$ varies with time according to the formulas [4.8]. As known from I, $\mathbf{l}$ comes as close as desired to each and every point of the unit square. Expressed pictorially, it may be said that the unit square becomes uniformly grey after a sufficiently long time. This feature is illustrated in figure 4(b).

The integral [4.7] may be transformed. In lieu of representing the pattern on the unit square, it may be equivalently represented on the parallelogram ABCD shown in figure 4(b). Sides AD and BC of this parallelogram are parallel to the trajectory of $\mathbf{l}$, i.e. their slopes are equal to ξ. The corresponding nonorthogonal coordinate system describing the location of a point within the parallelogram is denoted by α and β in figure 4(b).

The trajectory $\mathbf{l}$ intersects the x axis at successive instants t_n, separated by the constant time interval

$$\Delta t = 1/\xi. \tag{4.12}$$

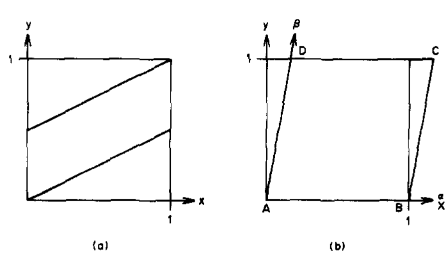

Figure 4. Illustration of the averaging process on the unit square for rational trajectories (a) and irrational trajectories (b).

Hence, the average [4.7] may be subdivided into the sum

$$\langle f \rangle = \lim_{N\to\infty} \frac{1}{N\Delta t} \sum_{n=0}^{N} \int_{t_n}^{t_{n+1}} f[\mathbf{l}(t)]\, \mathrm{d}t \qquad [4.13]$$

of contributions from successive intervals, where

$$t_{n+1} = t_n + \Delta t.$$

Denote by $\mathcal{L}(\alpha)$ the straight line

$$l_x = t + \alpha, \quad l_y = \xi t, \quad (\text{mod } 1). \qquad [4.14]$$

If $\alpha_n = n\xi$ is the abscissa of the trajectory at time t_n, [4.13] is easily transformed into the form

$$\langle f \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{\infty} X_{(\alpha_n)}, \qquad [4.15]$$

in which

$$X_{(\alpha_n)} = \int_{\mathcal{L}(\alpha_n)} f(\mathbf{l})\, \mathrm{d}s. \qquad [4.16]$$

The stage is now set to apply the basic ergodic theorem treated at length in I. According to [2.10] of that reference,

$$\langle f \rangle = \int_0^1 X(\alpha)\, \mathrm{d}\alpha.$$

Introduce [4.16] into this relation and perform the requisite change of variables to obtain

$$\langle f \rangle = \int_{[0,1]^2} f(\mathbf{l})\, \mathrm{d}x\, \mathrm{d}y. \qquad [4.17]$$

Expressed in words, the average value of f is equal to its integral over the unit square.

Apart from its striking simplicity, this result is remarkable in that it is independent of the precise value of ξ, as well as of the location l_{x0} of the origin. (For ξ rational, it was observed that just the opposite happened.) The present conclusion might have been anticipated on intuitive grounds. When ξ is irrational, the trajectory visits all the accessible positions with uniform probability; hence, one must obtain the same result whatever the precise mode (described by ξ and l_{x0}) of visitation.

These results will now be employed to calculate several interesting properties of suspensions undergoing simple shear.

4.3. *Explicit expression of the average in terms of the Fourier components*

The average [4.7] may be calculated in terms of the Fourier components of the function f in the following manner. Being a periodic function of $\mathbf{l}$, f may be expanded in the series

$$f = \sum_{\mathbf{m}} f_{\mathbf{m}} \exp(-2\pi i\, \mathbf{k}_{\mathbf{m}} \cdot \mathbf{l}). \qquad [4.18]$$

For the two-dimensional situation to which we are restricted, $\mathbf{k}_{\mathbf{m}}$ is given by

$$\mathbf{k}_{\mathbf{m}} = \frac{1}{\tau_o}(m_1 \mathbf{s}_1 + m_2 \mathbf{s}_2), \qquad [4.19]$$

with $\mathbf{s}_i$ ($i = 1, 2$) reciprocal basic lattice vectors of the unit square and vector $\mathbf{m}$ the diad $\{m_1, m_2\}$ of integers. Fourier components $f_\mathbf{m}$ appearing above are readily found to be given by the formula

$$f_\mathbf{m} = \int_{[0,1]^2} f \exp(2\pi i \mathbf{k}_\mathbf{m} \cdot \mathbf{l})\, ds. \qquad [4.20]$$

It has now become possible to calculate the various averages. When the coefficient ξ is irrational, it is found that according to [4.18] the average of f over the unit square is equal to the Fourier coefficient f_0. Explicitly,

$$\langle f \rangle = f_0 \qquad (\xi \text{ irrational}). \qquad [4.21]$$

When ξ is rational, and equal to p/q (where p and q are irreducible), introduction of [4.18] and [4.9] into [4.10] yields

$$\langle f \rangle = \frac{1}{q}\sum_\mathbf{m} f_\mathbf{m} \exp(-2\pi i m_1 l_{x0}) \int_0^q \exp\left[-2\pi i\left(m_1 + m_2\frac{p}{q}\right)t\right] dt \qquad [4.22]$$

after exchange of the summation and integration operations. Since the integrand in [4.22] is periodic (with a period q) its integral is equal to zero, except when

$$m_1 + m_2\frac{p}{q} = 0. \qquad [4.23]$$

Since p and q are irreducible, the set of solutions of this equation is

$$m_1 = np, \quad m_2 = -nq, \qquad [4.24]$$

when n is an integer. This implies the average [4.22] may be expressed as

$$\langle f \rangle = \sum_{n=-\infty}^{\infty} f_{\{np,-nq\}} \exp(-2\pi i n p l_{x0}), \quad (p \neq 0). \qquad [4.25]$$

For the sake of completeness, note that the corresponding expression for $p = 0$, corresponding to the case where $\mathbf{l}$ moves parallel to the x axis, is

$$\langle f \rangle = \sum_{m_2=-\infty}^{\infty} f_{\{0,m_2\}} \exp(-2\pi i m_2 l_{y0}), \quad (p = 0). \qquad [4.26]$$

Similarly, when $\mathbf{l}$ moves parallel to the y axis (which, loosely speaking, corresponds to $q = 0$),

$$\langle f \rangle = \sum_{m_1=-\infty}^{\infty} f_{\{m_1,0\}} \exp(-2\pi i m_1 l_{x0}). \qquad [4.27]$$

These calculations show that $\langle f \rangle$ displays a remarkable functional dependence upon the parameter ξ. It is constant for irrational values of ξ, whereas it is given by [4.25] for rational values. The appearance of this pathological type of function is certainly unexpected in deterministic fluid-mechanical problems.

An elementary example can be devised to illustrate these effects. Consider a Couette flow between corrugated walls, with the surfaces of the upper and lower plates respectively described by the equation

$$z_1 = 1 + \epsilon[\sin(x - x_o) + \sin(y - y_o)] \qquad [4.28]$$

and

$$z_2 = \epsilon(\sin x + \sin y). \qquad [4.29]$$

The upper plate will be assumed to move with a constant velocity **V** relative to the lower one. This velocity is, of course, parallel to the *x*-*y* plane. Coordinates (x_o, y_o) of a given point on the upper plate vary with time according to the expressions

$$\begin{aligned} x_o &= V_1 t + x_{oo}, \\ y_o &= V_2 t + y_{oo}. \end{aligned} \qquad [4.30]$$

Two parameters of physical interest here are the instantaneous force (per unit of wall surface area) required to maintain this uniform motion of the upper plate, and its time-averaged value. The calculation is analytically feasible for small values of the parameter ϵ, though it quickly leads to tedious algebra. No interesting effects are expected until terms of $O(\epsilon^2)$ are included, since the $O(\epsilon)$ term vanishes when spatially averaged. The second-order term however, remains, and contains expressions of the form $\exp(\pm ix_o)$ and $\exp(\pm iy_o)$. According to equations [4.25] to [4.27], two exceptional cases occur for special values of the integral. These correspond to

motion parallel to x or y. [4.31]

For the other values of ξ, the average is constant and equal to its ergodic value.

Retention of $O(\epsilon^3)$ contributions may be expected to yield terms of the forms $\exp(\pm 2ix_o)$, $\exp(\pm 2iy_o)$.

This subsection may be concluded by emphasizing the difference between temporal and spatial phenomena with regard to ergodicity. Consider a Poiseuille flow between the pair of fixed wavy walls defined by [4.28] and [4.29]. The average properties of such a flow will be *continuous* functions of the direction of the macroscopic pressure drop, in marked contrast to the situation just studied.

4.4. *Average properties of a suspension undergoing simple shear*

Identical external forces $\mathbf{F}^{(e)}$ and couples $\mathbf{N}^{(e)}$ will be assumed to act upon each of the particles in the suspension. The equations of motion [2.36] and [2.43], together with [3.8], imply that the average properties of the suspension are given by

$$\langle \mathbf{U}_0 - \bar{\mathbf{v}}^* \rangle = \mu^{-1} \langle {}^t\mathbf{K}^{-1} \rangle \cdot \mathbf{F}(e), \qquad [4.32a]$$

$$\langle \mathbf{\Omega} - \bar{\boldsymbol{\omega}} \rangle = \langle {}^r\mathbf{K}^{-1} \cdot \boldsymbol{\tau} \rangle : \bar{\mathbf{S}} + \mu^{-1} \langle {}^r\mathbf{K}^{-1} \rangle \cdot \mathbf{N}^{(e)}, \qquad [4.32b]$$

$$\langle \mathbf{A} \rangle = -\langle \mathbf{N}^* \cdot {}^r\mathbf{K}^{-1} \rangle \cdot \mathbf{N}^{(e)} - \mu \langle \mathbf{N}^* \cdot {}^r\mathbf{K}^{-1} \cdot \boldsymbol{\tau} - \mathbf{Q} \rangle : \bar{\mathbf{S}}. \qquad [4.32c]$$

These average properties may be explicitly calculated by the process described in the previous subsection, whence they may be regarded as known in principle. Recall that these averages will also depend upon the static elements of the kinematic description, as discussed in section 4.1.

Equation [4.32a] represents the average sedimentation velocity of the suspension when subjected to the external force $\mathbf{F}^{(e)}$. Though this sedimentation is not directly induced by the shear, its precise value nevertheless depends upon the shear—more precisely on the pattern $\mathcal{L}$ induced by the shear. This constitutes an interesting feature, in the sense that the phenomenon is susceptible to experimental measurement.

The average angular velocity of the particles (relative to the fluid) is more likely to prove an intermediate quantity in the calculations than a directly measured quantity itself. Nevertheless, the preceding comments about the sedimentation velocity still remain cogent.

Of course, in most rheological studies the quantity of major interest is the average particle stress, given by [4.32c], or a variant thereof. Though included for completeness, the external couple $\mathbf{N}^{(e)}$ will normally be absent in most circumstances (Brenner 1970 and Brenner & Weissman 1972). In such situations the particle stress is then given by the expression

$$\langle \mathbf{A} \rangle = -\mu \langle \mathbf{N}^* \cdot {}^{r}\mathbf{K}^{-1} \cdot \boldsymbol{\tau} - \mathbf{Q} \rangle : \bar{\mathbf{S}}. \qquad [4.33]$$

4.5. *Discussion*

Preceding results for the simple shear case are relevant in a variety of potential applications, in the sense that all the pertinent rheological properties may be derived in a simple but rigorous way. (The word "simple" does not, however, apply to the analytical and numerical efforts that must be expended to bring the calculations to fruition.) Once again, the essentially geometrical character of these dynamical results bears emphasis. This is, perhaps, not surprising in retrospect, since the kinematics have been reduced to a geometric evaluation, while the efficacy of the grand resistance matrix, which embodies the dynamical elements, stems from the basic fact that it depends only upon the geometrical configuration of the system. Obviously, these two independent geometrical elements supplement one another nicely, and together from a potent combination, useful perhaps in even more general circumstances.

The second important feature here is the ergodic character, or lack thereof, of the process, depending upon the rational or irrational nature of ξ. This leads inevitably to the fascinating question: "Does a real system choose between these values of ξ, and, if so, how?" It is interesting to note that the boundaries are neutral with respect to the choice of ξ whenever they are compatible with the flow; thus, for a slide flow the walls must be parallel to the slide, whereas for a tube flow they must be parallel to the tube. In both cases there remains an additional degree of freedom, which is precisely the choice of ξ. This is further illustrated in figure 5.

There exist many other examples of indeterminacy arising when inertia is neglected, such as the settling orientation of a homogeneous ellipsoidal particle in the Stokes regime (Cox 1965) or the Jeffery (1922) "orbit constant" of a neutrally buoyant spheroid undergoing rotation in a simple shear flow (Harper & Chang 1968). Hence, the above indeterminacy might similarly be removed by the inclusion of inertial effects. However, this delicate problem lies beyond the scope of the present Stokesian context. Probabilistic effects, such as Brownian motion, afford another possibility for removing the indeterminacy (Leal & Hinch 1971 and Hinch & Leal 1972).

Finally, hyperbolic and elliptic flows deserve at least a few comments. As shown in I, a suspension undergoing a hyperbolic flow does not reproduce itself in time. Accordingly, its time average is not physically meaningful, even if it is assumed to exist. Elliptic flow

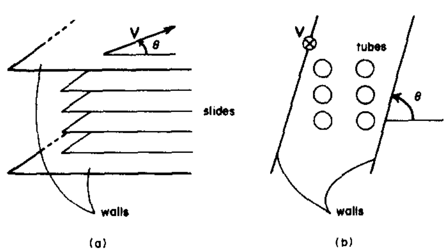

Figure 5. Boundaries and simple shear flow: (a) Slide flow. The slides are represented by the intermediate planes between the walls. The direction θ of the relative velocity **V** (parallel to both planes) of the two planes is arbitrary; (b) Tube flow. The view is edge on. The walls, the tube and the relative velocity **V** of the two walls are perpendicular to the plane of the figure. The direction θ of the walls is arbitrary, provided that the walls remain parallel to the tube axis.

represents perhaps the simplest case among the class of two-dimensional flows, since it is always self reproducing. However, its very simplicity is itself a course of disappointment, since it is a relatively easy matter to determine the configurations explored by the suspension during each period and to perform the time integrations along these configurations.

5. CONCENTRATED SUSPENSIONS AND LUBRICATION-THEORY TECHNIQUES

Application of "lubrication-theory" techniques to the rheology of concentrated suspensions was pioneered by Frankel & Acrivos (1967). Since the kinematical basis of their suspension model is *ad hoc,* rather than systematic, it is appropriate to briefly review their arguments. As the first step, the energy dissipation within the small gap between adjacent sphere pairs is calculated using the axial two-sphere solution (Brenner 1961) for two spheres approaching one another along their line of centers. Dissipation due to sliding motion between the two spheres in relative motion perpendicular to their line of centers is negligible when compared to the dissipation arising from the normal component of the motion. The relative velocity of the two spheres, as well as the time-average gap between them, are estimated from average values. All relative orientations are assumed equally likely, while the average gap width is deduced from the specified concentration, assuming a simple cubic arrangement of the spheres.

The following analysis outlines a rigorous approach to this same problem, taking account of the temporal evolution of the suspension configuration. Specifically, the general techniques developed in previous sections will be employed to analyze these concentrated suspensions at the lubrication-theory level of approximation. First, the kinematics will be detailed and a general expression obtained for the macroscopic stress tensor. Last, the averaging process will be discussed.

5.1. *Kinematics and forces in a linear chain*

Consider the motion of a linear chain of spheres, the centers of which are separated by the lattice vector $\mathbf{l}$ (figure 6). This situation is reminiscent of the one studied by Zia *et al.* (1967); however, additional sphere chains will eventually be added in order to form the complete lattice. Thus, the chains are here considered only for expediency, rather than as isolated geometrical entities of interest in their own right.

Let R and h, respectively, denote the radius of, and gap between, the spheres. From previous definitions there results

$$l = |\mathbf{l}| = 2R + h. \qquad [5.1]$$

Restriction to the asymptotic lubrication-theory limit,

$$\epsilon = h/R \ll 1, \qquad [5.2]$$

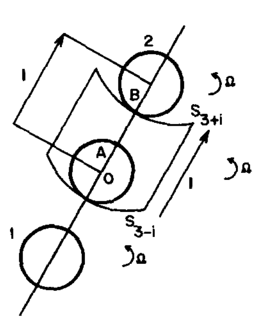

Figure 6. Linear chain of spheres.

thus implies that

$$|\mathbf{l}|/2R \simeq 1. \qquad [5.3]$$

Referring to figure 6, the velocity of point B relative to A may be expressed as

$$\mathbf{v}_B - \mathbf{v}_A = \mathbf{l} \cdot \mathbf{G} - 2\frac{R}{|\mathbf{l}|}(\boldsymbol{\Omega} \times \mathbf{l}), \qquad [5.4]$$

in which $\mathbf{G}$ and $\boldsymbol{\Omega}$ have their previous significance. Whereas $\mathbf{G}$ is specified *a priori*, $\boldsymbol{\Omega}$ must be determined as part of the solution. Introduction of [5.3] into [5.4] yields

$$\mathbf{v}_B - \mathbf{v}_A = \mathbf{l} \cdot \mathbf{G} - \boldsymbol{\Omega} \times \mathbf{l}. \qquad [5.5a]$$

Equivalently, with use of [2.24],

$$\mathbf{v}_B - \mathbf{v}_A = \mathbf{l} \cdot \overline{\mathbf{S}} - (\boldsymbol{\Omega} - \overline{\boldsymbol{\omega}}) \times \mathbf{l}. \qquad [5.5b]$$

Subsequent requirements necessitate decomposition of this relative velocity into its tangential and normal components, namely

$$\mathbf{v}_B - \mathbf{v}_A = \mathbf{l}(\hat{\mathbf{l}}\hat{\mathbf{l}}:\overline{\mathbf{S}}) + (\mathbf{I} - \hat{\mathbf{l}}\hat{\mathbf{l}}) \cdot \overline{\mathbf{S}} \cdot \mathbf{l} - (\boldsymbol{\Omega} - \overline{\boldsymbol{\omega}}) \times \mathbf{l}, \qquad [5.5c]$$

in which $\hat{\mathbf{l}} \equiv \mathbf{l}/l$.

Cox (1974) exhaustively treated the hydrodynamics of convex particles almost in contact via a rigorous, matched asymptotic form of lubrication theory. His results for two equal spheres may be displayed in tensorial form. Thus, the force may be expressed as

$$\mathbf{F} = -\mu\epsilon^{-1}(3\pi R/2)\mathbf{l}(\hat{\mathbf{l}}\hat{\mathbf{l}}:\overline{\mathbf{S}}) + \mu(\ln \epsilon)\pi R[\mathbf{l} \cdot \overline{\mathbf{S}} - \mathbf{l}(\hat{\mathbf{l}}\hat{\mathbf{l}}:\overline{\mathbf{S}}) - (\boldsymbol{\Omega} - \overline{\boldsymbol{\omega}}) \times \mathbf{l}], \qquad [5.6]$$

in which the first term reflects the effect of the normal motion, and the second the sliding motion.

The torque evaluated at point A is zero since the spheres possess the same angular velocity. However, the force [5.6] produces a torque

$$\mathbf{N}_o = (1/2)\mu(\ln \epsilon)\pi R\mathbf{l} \times [\mathbf{l} \cdot \overline{\mathbf{S}} - (\boldsymbol{\Omega} - \overline{\boldsymbol{\omega}}) \times \mathbf{l}] \qquad [5.7]$$

with respect to the center O of the sphere, wherein the 1/2 factor arises as a consequence of the asymptotic approximation [5.3].

5.2. *Equilibrium and macroscopic stress in a concentrated suspension*

The set of spheres adjacent to an arbitrary reference sphere centered at O is denoted by A. In turn, a sphere belonging to A is enumerated by the index i.

Since the suspension of spheres is centrally symmetric, the total force exerted by the adjacent spheres on the reference sphere is necessarily zero. But the total torque is not, as may be seen from [5.7]. The torque exerted by sphere 2 on the reference sphere at O is equal to that exerted by sphere 1 with respect to the same point. This results in the equilibrium condition

$$\sum_{i \in A} \mathbf{N}_{o,i} = \mathbf{0}. \qquad [5.8]$$

Introduction of [5.7] into [5.8], and subsequent division by $\mu\pi R/2$ yields

$$\sum_{i\in A}(\ln \epsilon_i)\,\mathbf{l}_i \times [\mathbf{l}_i \cdot \bar{\mathbf{S}} - (\boldsymbol{\Omega} - \bar{\boldsymbol{\omega}}) \times \mathbf{l}_i] = \mathbf{0}, \qquad [5.9]$$

a requirement that has previously been overlooked (Frankel & Acrivos 1967). It may be regarded as furnishing (from the prescribed data) the instantaneous value of the angular velocity $\boldsymbol{\Omega}$ with which the spheres rotate. Here, $\epsilon_i = h_i/R$.

The deviatoric part of the macroscopic stress tensor obtained from [2.50b] is

$$\bar{\mathbf{T}} = \frac{1}{\tau_o}\oint_{\partial\tau_o}\left(\mathbf{r}\mathbf{I} - \frac{1}{3}\mathbf{I}\mathbf{r}\right)\cdot d\mathbf{s}\cdot\breve{\mathbf{P}}. \qquad [5.10]$$

This integration is most readily effected by choosing the particular unit cell over which the integration is to be performed to be that depicted in figure 6. The dominant contribution to this integral arises from those regions lying within the narrow gaps. Moreover, the contributions of the several gaps are independent, and hence may be simply added. This sum may be further decomposed into the form

$$\bar{\mathbf{T}} = \frac{1}{\tau_o}\sum_{i\in A/2}\int_{s_{3+i}}\left(\mathbf{r}^+\mathbf{I} - \frac{1}{3}\mathbf{I}\mathbf{r}^+\right)\cdot d\mathbf{s}^+\cdot\breve{\mathbf{P}} + \int_{s_{3-i}}\left(\mathbf{r}^-\mathbf{I} - \frac{1}{3}\mathbf{I}\mathbf{r}^-\right)\cdot d\mathbf{s}^-\cdot\breve{\mathbf{P}}, \qquad [5.11]$$

where $A/2$ recalls that the summation is only taken over half of the adjacent spheres. However, at the geometrically equivalent points (Brenner 1980) $\mathbf{r}^+$ and $\mathbf{r}^-$ lying on the opposite faces (cf. figure 6) s_{3+i} and s_{3-i} of the unit cell,

$$d\mathbf{s}^+ = -d\mathbf{s}^-$$

and

$$\mathbf{r}^+ = \mathbf{r}^- + \mathbf{l}_i,$$

whose introduction into [5.11] yields

$$\bar{\mathbf{T}} = \frac{1}{\tau_o}\sum_{i\in A/2}\int_{s_{3+i}}\left(\mathbf{l}_i\mathbf{I} - \frac{1}{3}\mathbf{I}\mathbf{l}_i\right)\cdot d\mathbf{s}^+\cdot\breve{\mathbf{P}}.$$

Equivalently, since $\mathbf{l}_i$ is position independent,

$$\bar{\mathbf{T}} = \frac{1}{\tau_o}\sum_{i\in A/2}\left(\mathbf{l}_i\mathbf{I} - \frac{1}{3}\mathbf{I}\mathbf{l}_i\right)\cdot\int_{s_{3+i}} d\mathbf{s}^+\cdot\breve{\mathbf{P}}.$$

The singular part of the preceding integral represents the force $\mathbf{F}_i$ exerted by the fluid on the reference particle. Consequently,

$$\bar{\mathbf{T}} = \sum_{i\in A/2}\left(\mathbf{l}_i\mathbf{I} - \frac{1}{3}\mathbf{I}\mathbf{l}_i\right)\cdot\mathbf{F}_i.$$

For a given i, $\mathbf{F}_i$ is given by [5.6], whereupon

$$\bar{\mathbf{T}} = \frac{\mu\pi R}{\tau_o}\sum_{i\in A/2}[(\ln \epsilon_i)\mathbf{l}_i\{\mathbf{l}_i\cdot\bar{\mathbf{S}} - \mathbf{l}_i(\hat{\mathbf{l}}_i\hat{\mathbf{l}}_i{:}\bar{\mathbf{S}}) - (\boldsymbol{\Omega} - \bar{\boldsymbol{\omega}})\times\mathbf{l}_i\}$$
$$- (1/2)\epsilon_i^{-1}(\mathbf{l}_i\mathbf{l}_i{:}\bar{\mathbf{S}})\,(3\hat{\mathbf{l}}_i\hat{\mathbf{l}}_i - \mathbf{I})]. \qquad [5.12]$$

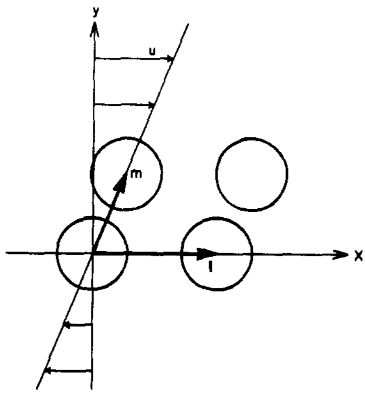

Figure 7. Two-dimensional array in a simple shear flow.

Observe that this relation does not depend upon the algebraic sign of $\mathbf{l}$, an expected conclusion. Thus, once again, the sum can be extended over A, rather than $A/2$, provided that a compensatory factor of $1/2$ is inserted. Since the total couple exerted by the fluid upon the particles is zero as in [5.9], the antisymmetric part of $\overline{\mathbf{T}}$ necessarily vanishes, as required by [2.57] and [2.60]. Hence, [5.12] may be expressed as

$$\overline{\mathbf{T}} = \frac{\mu\pi R}{2\tau_o} \operatorname{sym} \sum_{i \in A} [(\ln \epsilon_i)\{\mathbf{l}_i \cdot \overline{\mathbf{S}}\mathbf{l}_i - (\hat{\mathbf{l}}_i\hat{\mathbf{l}}_i{:}\overline{\mathbf{S}})\mathbf{l}_i\mathbf{l}_i - \mathbf{l}_i(\mathbf{\Omega} - \overline{\boldsymbol{\omega}}) \times \mathbf{l}_i\} \\ - (1/2)\epsilon_i^{-1}(\mathbf{l}_i\mathbf{l}_i{:}\overline{\mathbf{S}})\,(3\hat{\mathbf{l}}_i\hat{\mathbf{l}}_i - \mathbf{I})], \qquad [5.13]$$

in which the angular velocity $\mathbf{\Omega}$ is that obtained from [5.9], and in which $\operatorname{sym} \mathbf{D} = (1/2)(\mathbf{D} + \mathbf{D}\dagger)$ for any dyadic $\mathbf{D}$.

This general relationship [5.13], which appears not to have been given before, is inhomogeneous with respect to the orders of the various terms in ϵ_i involved; hence, it must be manipulated cautiously in order to avoid introducing inconsistencies regarding the order-of-magnitude accuracy of subsequent expressions. It provides a convenient compilation of the various component motion contributions. Observe that all quantities appearing in this expression are objective, as indeed they must be if the equation is to possess physical meaning.

5.3. *The averaging problem for a two-dimensional suspension in a simple shear flow*

Attention will be restricted in this section to performing the averaging operation over two- rather than three-dimensional suspensions. These display all relevant conceptual features, sans unnecessary complications. The pair of basic vectors required to characterize the lattice will be denoted $\mathbf{l}$ and $\mathbf{m}$, respectively (see figure 7); however, l will be assumed sufficiently large such that (cf. [5.3])

$$l - 2\mathrm{R} \simeq O(l), \qquad [5.14]$$

whence there exists no singularity associated with $\mathbf{l}$. Components of the vector $\mathbf{m}$ are

$$\mathbf{m}: \quad (x, 2R + \epsilon R). \qquad [5.15]$$

Expressed in component form, the simple shear velocity field is

$$u = Gy, \quad v = 0, \quad w = 0. \qquad [5.16]$$

Lubrication theory applies when x is small, corresponding to almost touching pairs of spheres in the lattice. Since $\Omega - \bar{\omega}$ possesses a component only in the z direction, its value may be obtained from [5.9] as

$$\Omega - \bar{\omega} = \hat{\mathbf{m}} \times (\hat{\mathbf{m}} \cdot \bar{\mathbf{S}}), \qquad [5.17]$$

with $\hat{\mathbf{m}} = \mathbf{m}/|\mathbf{m}|$. When the latter is introduced into [5.13], those terms involving ln ϵ vanish. This was to be expected, since the torque requirement [5.9] forbids any rotational slip in the situation under investigation. Hence, the macroscopic deviatoric stress reduces to

$$\bar{\mathbf{T}} = \frac{\mu\pi R}{2\tau_o}\epsilon^{-1}(\mathbf{mm}:\bar{\mathbf{S}})(3\,\hat{\mathbf{m}}\hat{\mathbf{m}} - \mathbf{I}). \qquad [5.18]$$

According to [4.11], the time-average stress may be computed from the formula

$$\langle\bar{\mathbf{T}}\rangle = \frac{1}{l}\int_{-l/2}^{l/2}\bar{\mathbf{T}}\,\mathrm{d}x. \qquad [5.19]$$

Each component of $\bar{\mathbf{T}}$ may be calculated as a function of the abscissa x. Evaluation of the singular portion of [5.19] is most naturally effected via introduction of an inner variable $\tilde{x}$, defined as

$$x = R\epsilon^{1/2}\tilde{x}. \qquad [5.20]$$

It is now easily verified that the resulting integral is *not* singular. In other words, terms of $O(1)$ result. And such terms are of the same order of magnitude as those deriving from the domain lying outside of the lubrication region, and which have thus been neglected in our calculations. Consequently, the present calculation must be terminated as being internally inconsistent. This disappointing conclusion is related to the fact that there exists no normal relative motion of the two spheres at $x = 0$.

It might be argued that the rotational slip terms may perhaps lead to singular contributions in situations where they do not vanish, in contrast to the current state of affairs. However, this argument is spurious, since the contribution of any typical factor of order ln ϵ_i in [5.13] may be confirmed to be nonsingular for the basic reason that the integral of ln x is finite in the neighborhood of $x = 0$, despite the fact that ln x itself tends to infinity at this point.

The preceding analysis leads to the paradoxical conclusion that the singular terms (if any) in concentrated suspensions cannot be derived by rigorous matched asymptotic, lubrication-theory-type arguments. For though the *instantaneous* stress itself tends to infinity in the touching-sphere limit, the *time-average* stress remains perfectly finite.

In this limit, other "nonhydrodynamic" factors (Cox & Brenner 1967) may predominate. Included are such potentially important features as surface roughness, cavitation, particle elasticity and lattice disorder, any one of which might give rise to singular behavior in the limit. However, it appears premature to embark upon a detailed discussion of the relative importance of such phenomena in the interpretation and rationalization of experimental results.

6. EXTENSIONS TO NONLINEAR FLOW PHENOMENA

Preceding results may be extended in a variety of directions. Section 6.1 concerns itself with non-Newtonian fluids, as being illustrative of such extensions, by focusing attention on two broad classes of such substances.

As has been emphasized in [2.29], the spatial periodicity of the pressure gradient is a basic prerequisite of the previous development. Any constitutive relationships that act to preserve this periodicity will permit retention of many of the prior results. This fact will be illustrated in section 6.2 via pertinent comments concerning inertial effects.

6.1. *Non-Newtonian fluids*

For Reiner-Rivlin fluids (cf. Aris 1962) the stress tensor is assumed to be of the generic constitutive form

$$\mathbf{P} = -\mathbf{I}p + \mathbf{T}(\mathbf{S}), \tag{6.1}$$

where $\mathbf{T}$ is any isotropic function of $\mathbf{S}$, not necessarily linear. Stokes' equation [2.3a] now adopts the more general form

$$\nabla p = -\nabla \cdot \mathbf{T}(\mathbf{S}). \tag{6.2}$$

Thus, for circumstances in which $\mathbf{S}$ is spatially periodic, this same periodic attribute obviously extends to the pressure gradient too. Major elements of prior developments remain valid in present circumstances, such as expressions [2.31] for the force, [2.40] for the couple and [2.48] for the macroscopic stress. Despite the fact that most of the prior formulas for the *instantaneous* properties of the suspension remain valid, the geometry and flow strength can, however, no longer be separated. Rather, any physical quantity g of interest will now appear as a nonlinear function,

$$g = \mathrm{g}(\mathbf{L}, \mathbf{G}), \tag{6.3}$$

of the geometry and of the macroscopic shear itself, in which the time does not explicitly appear (since the motion is assumed quasisteady).

When the suspension undergoes a simple shear $\mathbf{G}$, $\mathbf{L}$ becomes an almost periodic function of time. Since the motion is compatible (cf. I), g is necessarily a continuous function of $\mathbf{L}$, and thus an almost periodic function of time too. As such, according to the properties tabulated in section 4.2 of I, g possesses the time-average value

$$\langle g(\mathbf{L}, \mathbf{G}) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T g(\mathbf{L}(t), \mathbf{G})\, \mathrm{d}t, \tag{6.4}$$

which is independent of the time origin. Hence, the mean value $\langle g \rangle$ of any quantity g of physical interest is necessarily well defined.

Reiner-Rivlin fluids embrace a rather large class of substances, which can be still further enlarged by considering fluids for which the stress tensor depends upon the deformation history. Though a detailed account of this case will not be presented here, it is nevertheless useful to highlight some aspects of the flow problem which were not emphasized before.

First of all, apart from being spatially periodic, the velocity field needs to be assumed almost periodic in time. This latter characteristic is, of course, obvious for a Newtonian fluid, as a direct consequence of the almost periodicity of the suspension geometry. Hence, the stress tensor—which is a function of the deformation history—must also be almost periodic in time, provided that the functional entails functions which transmit the almost periodic character (cf. section 4.2 of I). In this event, the previous scheme can be followed, and any property g of physical interest computed.

This case obviously poses the most difficulties, since no separation now exists among the three basic ingredients of the problem: history, flow strength and geometry.

6.2. *Inertial effects*

Thus far, inertial effects have been supposed negligible. Their inclusion yields the complete Navier-Stokes equations,

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v}, \qquad [6.5]$$

with D/Dt the material derivative. Two limiting situations may be distinguished. In the first, the quasistatic flow approximation holds, whence the preceding reduces to

$$\rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v}. \qquad [6.6]$$

A spatially periodic velocity field thereby implies a spatially periodic pressure gradient. This corresponds, for instance, to Darcy flow through a porous medium, a topic that has been extensively studied elsewhere (Brenner & Adler 1985). However, when the velocity field is not spatially periodic, as happens in the case of sheared suspensions, ∇p is no longer spatially periodic, whence the present analysis ceases to be applicable.

The case where unsteady local acceleration terms in [6.5] predominate over convective acceleration terms may be of interest, corresponding to the equation

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mu \nabla^2 \mathbf{v}. \qquad [6.7]$$

This relation is tractable in the present context, since the quintessential linear character of the problem is retained. Various fields of interest (or their gradients) are now spatially periodic and/or almost periodic in time, whence the present mode of analysis may be brought to bear upon their resolution in particular circumstances.

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NOMENCLATURE†

A	set of spheres immediately adjacent to an arbitrary reference sphere
A,B	points on opposite sides of the gap between spheres in figure 6
$\mathbf{C}$, C_{ij}	hydrodynamic coupling pseudodyadic of a particle
D/Dt	material derivative
e	external
f	fluid, or function of arbitrary tensorial rank dependent only upon suspension geometry
$\langle f \rangle$	time-average value of the function f, defined in equation [4.7]
$\mathbf{f_m}$, $\mathbf{f}_{\{m_1,m_2\}}$	Fourier coefficient defined by equation [4.20]
$\mathbf{F}$	hydrodynamic force on particle(s) in cell $\{\mathbf{n}\}$
$\mathbf{F}^{(e)}$	external force exerted upon particle(s) in cell $\{\mathbf{n}\}$
$\overline{\mathbf{F}}$	hydrodynamic force density defined in equation [2.30]
$g(\mathbf{L}, \mathbf{G})$	function of instantaneous suspension geometry and shear
h	gap between adjacent spheres
$\breve{\mathbf{h}}$	generic spatially periodic field
i	ith sphere
$\mathbf{I}$	dyadic idemfactor

†Only new symbols, not already introduced in Part I, are defined here.

$\mathbf{k_m}$	discrete vector defined by equation [4.19] in reciprocal lattice vector space
$^r\mathbf{K}$, $^rK_{ij}$	hydrodynamic rotational resistance dyadic of a particle
$^t\mathbf{K}$, $^tK_{ij}$	hydrodynamic translational resistance dyadic of a particle
l	characteristic linear dimension of unit cell or magnitude $\lvert \mathbf{l} \rvert$ of $\mathbf{l}$
l_x, l_y	components of the vector $\mathbf{l}$
l_{x0}	value of l_x at time $t = 0$
$\mathbf{l}$	time-dependent vector defined in equation [4.5] or center-to-center vector drawn between adjacent sphere centers
$\mathbf{l}_i$	basic lattice vector
$\hat{\mathbf{l}}_i$	unit basic lattice vector
L	characteristic linear dimension of suspension appearing in equation [2.16], or length of "pattern" in equation [4.11]
$\mathcal{L}$	"intermediate" length scale defined in equation [2.16], or pattern made by $\mathbf{l}$ on the unit square, as in figure 4
$\mathcal{L}(\alpha)$	straight line defined in equation [4.14]
m	magnitude of basic lattice vector $\mathbf{m}$
m_1, m_2	integers
$\mathbf{m}$	diad $\{m_1, m_2\}$ of integers, or basic lattice vector in figure 7
$\mathbf{M}$, M_{ijk}	Stokeslet-slip velocity hydrodynamic resistance triadic
$\mathbf{n}$ or $\{\mathbf{n}\}$	cell $\mathbf{n}$ or $\mathbf{n}$th particle
N	upper summation limit in equation [4.13]
$\mathbf{N}$	hydrodynamic couple
$\mathbf{N}_o$	hydrodynamic torque about the center O of a sphere
$\mathbf{N}^{(e)}$	external couple on a particle
$\overline{\mathbf{N}}^e$	external couple density defined in equation [2.61]
$\mathbf{N}^*$, N^*_{ijk}	Stokeslet-angular velocity hydrodynamic resistance pseudotriadic
$\mathcal{N}$	number of unit cells contained in the intermediate volume $\mathcal{V}$
o or O	sphere center, origin
$O(l)$	gage symbol expressing an eror of order of the cell size l
$O_\mathbf{n}$	origin at the center of the $\mathbf{n}$th particle or unit cell
p	local pressure field, or particle
$\bar{p}$	macroscopic or bulk pressure
$\breve{p}(\mathbf{R})$	spatially periodic portion of local pressure field at point $\mathbf{R}$
$\overline{\mathbf{P}}_\times$	vector invariant of macroscopic stress field $\overline{\mathbf{P}}$, defined in equation [2.57]
$\mathbf{P}$	local stress field
$\breve{\mathbf{P}}(\mathbf{R})$	spatially periodic portion of local stress field at point $\mathbf{R}$
$\overline{\mathbf{P}}$	macroscopic stress field in the suspension defined in equation [2.45]
$\mathbf{Q}$, Q_{ijkl}	Stokeslet rate of strain tetradic
$\mathbf{r}$	local position vector defined in equation [2.7]
$\mathbf{r}^+$, $\mathbf{r}^-$	local position vectors at equivalent points lying on the opposite faces of a unit cell
R	radius of sphere
$\overline{\mathbf{R}}$	position vector of the center of volume of a domain τ
$d^3\mathbf{R}$	differential volume element
ds	differential element of length on the pattern $\mathcal{L}$
s_p, $s_p\{\mathbf{n}\}$	particle surface(s) within cell $\{\mathbf{n}\}$
$d\mathbf{s}$	directed element of surface area
$d\mathbf{s}^+$, $d\mathbf{s}^-$	directed surface elements lying at equivalent points on opposite cell faces
$\mathbf{s}_i$	ith reciprocal basic lattice vector

s_{3+i}, s_{3-i} opposite faces of a unit cell, as shown in figure 6
S closed surface bounding a volume τ
$\mathbf{S}$ local rate-of-strain dyadic defined in eqn [2.4]
$\bar{\mathbf{S}}$ macroscopic rate-of-strain dyadic for the suspension, defined in equation [2.24]
t_n end of the nth time interval
Δt time interval defined in equation [4.12]
$\mathbf{T}$ deviatoric stress dyadic
$\bar{\mathbf{T}}$ mean deviatoric stress dyadic in the suspension, defined in equation [2.49]
$\mathbf{U}_0$ velocity of the particle in the zeroth cell
$\mathbf{U}_\mathbf{n}$ velocity of the particle in the $\mathbf{n}$th cell
$\mathbf{v}$ local fluid velocity vector field
$\check{\mathbf{v}}$ spatially periodic portion of local fluid velocity vector field defined in equation [2.26], and having a zero mean value
$\check{\mathbf{v}}'$ spatially periodic portion of local fluid velocity vector field defined in equation [2.11]
$\bar{\mathbf{v}}$ mean vector velocity field of suspension defined in equation [2.18]
$\bar{\mathbf{v}}^*$ mean interstitial vector velocity field of suspension, defined in equation [2.19]
$\mathbf{V}$ velocity vector of a plate
$\mathcal{V}$ intermediate volume domain
$\partial\mathcal{V}$ surface bounding the intermediate volume $\mathcal{V}$
$\mathcal{V}_f, \mathcal{V}_p$ fluid and particle domains of the intermediate volume $\mathcal{V}$
$\tilde{x}$ inner x coordinate defined in equation [5.20]
$z_1(x, y), z_2(x, y)$
equations of the upper and lower wavy walls bounding a Couette flow

Greek Letters

α,β nonorthogonal coordinate system defining point within a parallelogram in figure 4(b)
α_n abscissa of the $\mathbf{l}$ trajectory at time t_n ($\alpha_n = n\xi$)
ϵ dimensionless gap width defined in equation [5.2]
$\boldsymbol{\epsilon}$ unit alternating isotropic triadic
$\bar{\boldsymbol{\Lambda}}$ antisymmetric portion of the mean shear rate of the suspension, defined in equation [2.24]
μ viscosity of interstitial fluid
ρ density of interstitial fluid
τ volumetric domain in equation [2.15]
$\tau_f, \tau_f\{\mathbf{n}\}$ interstitial fluid volume contained within cell $\{\mathbf{n}\}$
$\partial\tau_o, \partial\tau_o\{\mathbf{n}\}$ outer surface of the $\mathbf{n}$th unit cell $\tau_o\{\mathbf{n}\}$
$\boldsymbol{\tau}, \tau_{ijk}$ torque-angular slip velocity hydrodynamic resistance triadic
$\bar{\boldsymbol{\tau}}$ symmetric portion of the macroscopic stress dyadic, defined in equation [2.51]
$\boldsymbol{\phi}, \phi_{ijk}$ force-shear rate hydrodynamic resistance triadic
$\Phi, \bar{\Phi}$ local and mean rates of energy dissipation, defined in equations [2.62] and [2.64]
$\bar{\boldsymbol{\omega}}$ mean angular velocity pseudovector of the suspension, defined by equation [2.25]
$\boldsymbol{\Omega}, \boldsymbol{\Omega}_\mathbf{n}$ angular velocity pseudovector of the particle in the $\mathbf{n}$th cell

Special Symbols

$\breve{}$ spatially periodic function
— suspension- or mean Darcy-scale value
$\langle\ \rangle$ time-average value defined in equation [4.7]
$\tilde{}$ inner stretched variable

REFERENCES

ADLER, P. M. 1981 Interaction of unequal spheres I. Hydrodynamic interaction. Colloidal forces. *J. Colloid Interface Sci.* 84, 461–474.

ADLER, P. M. & BRENNER, H. 1985 Spatially periodic suspensions of convex particles in linear shear flow I. Description and kinematics. *Int. J. Multiphase Flow* **11,** 361–385.

ARIS, R. 1962 *Vectors, tensors, and the basic equations of fluid mechanics.* Prentice-Hall, Englewood Cliffs, New Jersey.

ARNOLD, V. I. & AVEZ, A. 1968 *Ergodic problems of classical mechanics.* Benjamin, New York.

BENSOUSSAN, A., LIONS, J. L. & PAPANICOLAU, G. 1978 *Asymptotic analysis for periodic structures.* North-Holland, Amsterdam, New York and Oxford.

BRENNER, H. 1961 The slow motion of a sphere through a viscous fluid towards a plane surface. *Chem. Engng Sci.* 16, 242–251.

BRENNER, H. 1970 Rheology of two-phase systems. *Ann. Rev. Fluid Mech.* 2, 137–176.

BRENNER, H. 1974 Rheology of a dilute suspension of axisymmetric Brownian particles. *Int. J. Multiphase Flow* 1, 195–341.

BRENNER, H. 1980 Dispersion resulting from flow through spatially periodic porous media. *Phil. Trans. R. Soc. Lond.* A297, 81–133.

BRENNER, H. 1981 Taylor dispersion in systems of sedimenting nonspherical Brownian particles II. Homogeneous, ellipsoidal particles. *J. Colloid Interface Sci.* 80, 548–588.

BRENNER, H. & ADLER, P. M. 1985 *Transport processes in porous media.* Hemisphere/McGraw-Hill, New York.

BRENNER, H. & O'NEILL, M. E. 1972 On the Stokes resistance of multiparticle systems in a linear shear field. *Chem. Engng Sci.* 27, 1421–1439.

BRENNER, H. & WEISSMAN, M. H. 1972 Rheology of a dilute suspension of dipolar spherical particles in an external field. II. Effects of rotary Brownian motion. *J. Colloid Interface Sci.* 41, 499–531.

COX, R. G. 1965 The steady motion of a particle of arbitrary shape at small Reynolds number. *J. Fluid Mech.* 23, 625–643.

COX, R. G. 1974 The motion of suspended particles almost in contact. *Int. J. Multiphase Flow* 1, 343–371.

COX, R. G. & BRENNER, H. 1967 The slow motion of a sphere through a viscous fluid towards a plane surface. Part II. Small gap widths, including inertial effects. *Chem. Engng Sci.* 22, 1753–1777.

DAHLER, J. S. & SCRIVEN, L. E. 1963 Theory of structured continua: I. General consideration of angular momentum and polarization. *Proc. R. Soc. Lond.* A275, 504–527.

FRANKEL, N. A. & ACRIVOS, A. 1967 On the viscosity of a concentrated suspension of solid spheres. *Chem. Engng Sci.* 22, 847–853.

FRISCH, H. L. & SIMHA, R. 1956 The viscosity of colloidal suspensions and macromolecular solutions. *Rheology: Theory and applications* (edited by F. R. Eirich), vol. 1, pp. 525–613. Academic Press, New York.

HAPPEL, J. & BRENNER, H. 1965 *Low Reynolds number hydrodynamics.* Prentice-Hall, Englewood Cliffs, New Jersey.

HARPER, E. Y. & CHANG, I-DEE 1968 Maximum dissipation resulting from lift in a slow viscous shear flow. *J. Fluid Mech.* 33, 209–225.

HINCH, E. J. 1972 A remark on the symmetry of certain material tensors for a particle in Stokes flow. *J. Fluid Mech.* 54, 423–425.

HINCH, E. J. & LEAL, L. G. 1972 The effect of Brownian motion on the rheological properties of a suspension of non-spherical particles. *J. Fluid Mech.* 52, 683–712.

JEFFERY, D. J. & ACRIVOS, A. 1976 The rheological properties of suspensions of rigid particles. *Am. Inst. Chem. Engrs. J.* 22, 417–432.

JEFFERY, G. B. 1922 The motion of ellipsoidal particles immersed in a viscous fluid. *Proc. R. Soc. Lond.* A102, 161–179.

LAMB, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

LANDAU, L. D. & LIFSHITZ, E. M. 1958 *Statistical physics.* Addison-Wesley, Reading, Massachusetts.

LEAL, L. G. & HINCH, E. J. 1971 The effect of weak Brownian rotations on particles in shear flow. *J. Fluid Mech.* 46, 685–703.

RUTGERS, R. 1962 Relative viscosity and concentration. *Rheol. Acta* 2, 305–348.

THOMAS, D. G. 1965 Transport characteristics of suspensions VIII. A note on the viscosity of Newtonian suspensions of uniform spherical particles. *J. Colloid Sci.* 20, 267–277.

TRUESDELL, C. & NOLL, W. 1965 The nonlinear field theories of mechanics. *Handbuch der Physik* (edited by S. Flügge), vol. III/3. Springer-Verlag, Berlin, Heidelberg and New York.

ZIA, I. Y. Z., COX, R. G. & MASON, S. G. 1967 Ordered aggregates of particles in shear flow. *Proc. R. Soc. Lond.* A300, 421–441.